DIFFERENTIAL SYSTEMS ASSOCIATED WITH TABLEAUX OVER LIE ALGEBRAS

EMILIO MUSSO* AND LORENZO NICOLODI†

Abstract. We give an account of the construction of exterior differential systems based on the notion of tableaux over Lie algebras as developed in [33]. The definition of a tableau over a Lie algebra is revisited and extended in the light of the formalism of the Spencer cohomology; the question of involutiveness for the associated systems and their prolongations is addressed; examples are discussed.

Key words. Exterior differential systems, Pfaffian differential systems, involutiveness, tableaux, tableaux over Lie algebras.

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1. Introduction. The search for a common structure to various exterior differential systems (EDSs) of geometric and analytic origin led to the algebraic notion of a *tableaux over a Lie algebra* [33]. This notion builds on that of involutive tableau in the theory of EDSs and can be seen as a non-commutative generalization of it. Interestingly enough, from a tableau over a Lie algebra we can canonically construct a linear Pfaffian differential system (PDS) which is in involution and whose Cartan characters coincide with the characters of the tableau.

Particular cases of this scheme lead to differential systems describing well-known integrable systems such as the Grassmannian systems of Terng [37, 4], the curved flat system of Ferus and Pedit [16], and many integrable surfaces arising in projective differential geometry [1, 14, 17]. The tableaux corresponding to Grassmannian and curved flat systems, the so-called Cartan tableaux, are obtained from the Cartan decompositions of semisimple Lie algebras. The tableaux corresponding to the various classes of integrable surfaces in projective 3-space are given by sub-tableaux of a special tableau over $\mathfrak{sl}(4,\mathbb{R})$. This is constructed by the method of moving frames and amounts to the construction of a canonical adapted frame along a generic surface in projective space (e.g., the Wilczynski–Cartan frame [11, 17, 2, 14, 15]). The involutiveness of these examples follows from the involutiveness of the corresponding tableaux. The result that the Grassmannian system and the curved flat system are in involution was first proved by Bryant in a cycle of seminars at MSRI [6] and was then taken up and further elaborated by Terng and Wang [38].

Other examples of linear Pfaffian system in involution which fit into the above scheme, and in fact motivated our work, include: the differential systems of isothermic surfaces in Möbius and Laguerre geometry [25, 26, 28, 30]; the

* Dipartimento di Matematica Pura ed Applicata, Università degli Studi dell'Aquila, Via Vetoio, I-67010 Coppito (L'Aquila), Italy (musso@univaq.it). Partially supported by MIUR project *Metriche riemanniane e varietà differenziali*, and by the GNSAGA of INDAM.

† Dipartimento di Matematica, Università degli Studi di Parma, Viale G. P. Usberti 53/A - Campus universitario, I-43100 Parma, Italy (lorenzo.nicolodi@unipr.it). Partially supported by MIUR project *Proprietà geometriche delle varietà reali e complesse*, and by the GNSAGA of INDAM.

differential systems of Möbius-minimal (M-minimal or Willmore) surfaces and of Laguerre-minimal (L-minimal) surfaces [27, 29, 25]; the differential systems associated to the deformation problem in projective geometry and in Lie sphere geometry [11, 10, 3, 22, 32, 13]. The tableaux associated with all these examples are constructed by the method of moving frames on the Lie algebras of the corresponding symmetry groups (cf. Section 5.3). If, on the one hand, the presented approach may be seen as a possibility to discuss various involutive systems from a unified viewpoint, on the other hand, it can be viewed as a possibility to find new classes of involutive systems. In this respect, we mention the class of projective surfaces introduced in [33], which generalizes asymptotically-isothermic surfaces and surfaces with constant curvature of Fubini's quadratic form. An analogous class of surfaces in the context of conformal geometry is discussed in Section 5.3. For an application of the above construction to the study of the Cauchy problem for the associated systems, we refer the reader to [31, 34].

In this article we revisit the definition of tableau over a Lie algebra using the formalism of the Spencer cohomology and extend it to include 2-acyclic tableaux. This allows also non-involutive systems into the scheme, reducing the question of involutiveness of their prolonged systems to that of the prolongations of the associated tableaux (cf. Section 4).

Section 2 contains the basic material about tableaux. Section 3 introduces the notion of tableaux over Lie algebras. Section 4 presents the construction of EDSs from tableaux over Lie algebras and discusses some properties of such systems. Section 5 discusses some examples as an illustration of the theory developed in the previous sections. Further developments are indicated in Section 6. The appendix collects some facts about the Spencer complex of a tableau and the torsion of a PDS.

2. Tableaux. In this section we provide a summary of the results in the algebraic theory of tableaux. As basic reference, we use the book by Bryant, *et al.* [7]. See also [21].

A **tableau** is a linear subspace $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$, where $\mathfrak{a}$, $\mathfrak{b}$ are (real or complex) finite dimensional vector spaces.

An h-dimensional subspace $\mathfrak{a}_h \subset \mathfrak{a}$ is called *generic* w.r.t. $\mathbf{A}$ if the dimension of

$$\mathrm{Ker}\,(\mathbf{A}, \mathfrak{a}_h) := \{Q \in \mathbf{A} \mid Q_{|_{\mathfrak{a}_h}} = 0\}$$

is a minimum. The set of h-dimensional generic subspaces is a Zariski open of the Grassmannian of h-dimensional subspaces of $\mathfrak{a}$.

A flag $(0) \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{a}$ of $\mathfrak{a}$ is said *generic* if $\mathfrak{a}_h$ is generic, for all $h = 1, \ldots, n$.

The **characters** of $\mathbf{A}$ are the non-negative integers $s_j(\mathbf{A})$, $j = 1, \ldots, n$, defined inductively by

$$s_1(\mathbf{A}) + \cdots + s_j(\mathbf{A}) = \mathrm{codim}\,\mathrm{Ker}\,(\mathbf{A}, \mathfrak{a}_j),$$

for any generic flag $(0) \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{a}$.

From the definition, it is clear that

$$\dim \mathfrak{b} \geq s_1 \geq s_2 \geq \cdots \geq s_n, \quad \dim \mathbf{A} = s_1 + \cdots + s_n.$$

If $s_\nu \neq 0$, but $s_{\nu+1} = 0$, we say that $\mathbf{A}$ has *principal character* s_ν and call ν the *Cartan integer* of $\mathbf{A}$.

The **first prolongation** $\mathbf{A}^{(1)}$ of $\mathbf{A}$ is the kernel of the linear map (Spencer coboundary operator, cf. Appendix A)

$$\delta^{1,1} : \mathrm{Hom}(\mathfrak{a}, \mathbf{A}) \cong \mathbf{A} \otimes \mathfrak{a}^* \to \mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)$$

$$\delta^{1,1}(F)(A_1, A_2) := \frac{1}{2}\left(F(A_1)(A_2) - F(A_2)(A_1)\right),$$

for $F \in \mathrm{Hom}(\mathfrak{a}, \mathbf{A})$, and $A_1, A_2 \in \mathfrak{a}$.

The **h-th prolongation** of $\mathbf{A}$ is defined inductively by setting

$$\mathbf{A}^{(h)} = \mathbf{A}^{(h-1)^{(1)}},$$

for $h \geq 1$ (by convention $\mathbf{A}^{(0)} = \mathbf{A}$ and $\mathbf{A}^{(-1)} = \mathfrak{b}$). $\mathbf{A}^{(h)}$ identifies with

$$\mathbf{A}^{(h)} = \left(\mathbf{A} \otimes S^h(\mathfrak{a}^*)\right) \cap \left(\mathfrak{b} \otimes S^{h+1}(\mathfrak{a}^*)\right).$$

An element $Q_{(h)} \in \mathfrak{b} \otimes S^{h+1}(\mathfrak{a}^*)$ belongs to $\mathbf{A}^{(h)}$ if and only if $i(X)Q_{(h)} \in \mathbf{A}^{(h-1)}$, for all $X \in \mathfrak{a}$.

THEOREM 2.1. $\dim \mathbf{A}^{(1)} \leq s_1 + 2s_2 + \cdots + ns_n$.

DEFINITION 2.1. *$\mathbf{A}$ is said* ***involutive*** *(or* ***in involution****) if equality holds in the previous inequality.*

THEOREM 2.2. *For any tableau $\mathbf{A}$ there exists an integer h_0 such that $\mathbf{A}^{(h)}$ is involutive, for all $h \geq h_0$.*

THEOREM 2.3. *If $\mathbf{A}$ is involutive, then $\mathbf{A}^{(1)}$ is involutive and*

$$s_j^{(1)} := s_j(\mathbf{A}^{(1)}) = s_n(\mathbf{A}) + \cdots + s_j(\mathbf{A}),$$

$j = 1, \ldots, n$.

Thus *every prolongation of an involutive tableau is involutive*. Moreover, *the principal character and the Cartan integer are invariant under prolongation of an involutive tableau*.

It is well-known that $\mathbf{A}$ is involutive if and only if

$$H^{q,p}(\mathbf{A}) = (0), \quad \text{for all} \quad q \geq 1, p \geq 0$$

(Guillemin-Sternberg, Serre [20]). See Appendix A for the definition of the Spencer groups $H^{q,p}(\mathbf{A})$.

A weaker notion is the following.

DEFINITION 2.2. *A tableau* $\mathbf{A}$ *is said 2-**acyclic** if*

$$H^{q,2}(\mathbf{A}) = (0), \quad \textit{for all} \quad q \geq 1.$$

This notion plays an essential role in the prolongation procedure of a non-involutive linear Pfaffian system (cf. Kuranishi, Goldschmidt [18, 19]).

As shown in the following examples, tableaux and their prolongations arise naturally in the context of PDE systems and of exterior differential systems.

EXAMPLE 1. Let V and W be vector spaces with coordinates $x^1, \dots, x^n$ and $y^1, \dots, y^s$ dual to bases $v_1, \dots, v_n$ for V and $w_1, \dots, w_s$ for W. Consider the first-order constant coefficient system of PDEs for maps $f : V \to W$ given in coordinates by

$$B_a^{\lambda i} \frac{\partial y^a}{\partial x^i}(x) = 0 \quad (\lambda = 1, \dots, r). \tag{2.1}$$

The linear solutions $y^a(x) = A^a_j x^j$ to this system give rise to a tableau $\mathbf{A} \subset \mathrm{Hom}(V, W)$. Conversely, any tableau $\mathbf{A} \subset \mathrm{Hom}(V, W)$ determines a PDE system of this type.

REMARK 2.1. $\mathbf{A}^{(q)}$ *is the set of homogeneous polynomial solutions of degree* $q + 1$ *to* (2.1)*.*

THEOREM 2.4. *The PDE system associated to* $\mathbf{A}$ *is involutive* $\iff$ $\mathbf{A}$ *is involutive.*

The *symbol* of (2.1) is the annihilator $\mathbf{B} := \mathbf{A}^\perp \subset V \otimes W^*$ of $\mathbf{A}$.

EXAMPLE 2. Let $(\mathcal{I}, \omega)$ be a Pfaffian differential system (PDS) on a manifold M with independence condition $\omega \neq 0$, where

$$\mathcal{I} = \{\theta^1, \dots, \theta^s, d\theta^1, \dots, d\theta^s\} \quad \text{(algebraic ideal)}$$

and $\omega = \omega^1 \wedge \cdots \wedge \omega^n$. Let $\pi^1, \dots, \pi^t$ be 1-forms such that

$$\theta^1, \dots, \theta^s;\ \omega^1, \dots, \omega^n;\ \pi^1, \dots, \pi^t$$

be a local adapted coframe of M.

The Pfaffian differential system $(\mathcal{I}, \omega)$ is called **linear**[1] if and only if

$$d\theta^a \equiv 0 \mod \{\theta^1, \dots, \theta^s, \omega^1, \dots, \omega^n\} \quad (0 \leq a \leq s).$$

The meaning of this condition is that the variety $V_n(\mathcal{I}, \omega) \subset G_n(TM, \omega)$ of integral elements of $(\mathcal{I}, \omega)$ is described by a system of inhomogeneous linear equations (cf. Appendix B). A linear PDS is described locally by

$$\begin{cases} \theta^a = 0 \\ d\theta^a \equiv A^a_{\epsilon i} \pi^\epsilon \wedge \omega^i + \frac{1}{2} c^a_{ij} \omega^i \wedge \omega^j \quad \mod \{\theta^1, \dots, \theta^s\} \\ \omega = \omega^1 \wedge \cdots \wedge \omega^n \neq 0, \end{cases}$$

[1] In the literature, other names are also used to indicate linear systems: quasi-linear systems, systems in good form, or systems in normal form.

where $c^a_{ij} = -c^a_{ji}$; $1 \leq a \leq s$; $1 \leq i, j \leq n$; $1 \leq \epsilon \leq t$.

Once we fix independent variables and take a point of M, we can associate a tableau to the Pfaffian system as follows. At $x \in M$, let $V^* = \operatorname{span}\{\omega^i\}$ and $\{\frac{\partial}{\partial\omega^i}\}$ be the basis of its dual V. Further, let $W^* = \operatorname{span}\{\theta^a\}$ and $\{\frac{\partial}{\partial\theta^a}\}$ be the basis of its dual W. We define a tableau $\mathbf{A} \subset W \otimes V^*$ by

$$\mathbf{A} := \operatorname{span}\{A^a_{\epsilon i}\frac{\partial}{\partial\theta^a} \otimes \omega^i \,:\, \epsilon = 1, \ldots, t\}.$$

The involutiveness of $(\mathcal{I}, \omega)$ at x is equivalent to the involutiveness of $\mathbf{A}$ (algebraic condition) together with the integrability condition $V(\mathcal{I}, \omega)_{|x} \neq \emptyset$, which in turn is equivalent to the condition $c^a_{ij}(x) = 0$, for each a, i, j ("torsion vanishes at x"). See Appendix B for more on the notion of torsion.

THEOREM 2.5. *The linear PDS $(\mathcal{I}, \omega)$ is involutive at $x \iff$ 1) $\mathbf{A}$ is involutive and 2) $V_n(\mathcal{I}, \omega)_{|x} \neq \emptyset$.*

The *symbol* of $(\mathcal{I}, \omega)$ is the annihilator $\mathbf{B} := \mathbf{A}^\perp \subset V \otimes W^*$ of $\mathbf{A}$:

$$B = \operatorname{span}\{B^\lambda = B^{\lambda i}_a \frac{\partial}{\partial\omega^i} \otimes \theta^a : B^{\lambda i}_a A^a_{\epsilon i} = 0, \forall\lambda, \epsilon\}.$$

3. Tableaux over Lie algebras. Let $(\mathfrak{g}, [\,,])$ be a finite dimensional Lie algebra, $\mathfrak{a}, \mathfrak{b}$ vector subspaces of $\mathfrak{g}$ such that $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{g}$, and $\mathbf{A} \subset \operatorname{Hom}(\mathfrak{a}, \mathfrak{b})$ a tableau. Define the polynomial map $\tau : \mathbf{A} \to \mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)$ by

$$\begin{aligned}\tau(Q)(A_1, A_2) := {} & [A_1 + Q(A_1), A_2 + Q(A_2)]_{\mathfrak{b}} \\ & - Q\left([A_1 + Q(A_1), A_2 + Q(A_2)]_{\mathfrak{a}}\right),\end{aligned}$$

where $X_{\mathfrak{a}}$ (resp. $X_{\mathfrak{b}}$) denotes the $\mathfrak{a}$ (resp. $\mathfrak{b}$) component of X.

DEFINITION 3.1 ([33]). *A* ***tableau over*** $\mathfrak{g}$ *is a tableau $\mathbf{A} \subset \operatorname{Hom}(\mathfrak{a}, \mathfrak{b})$ such that:*

1. *$\mathbf{A}$ is involutive;*
2. *$\tau(Q) \in Im\, \delta^{1,1} \subset \mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)$, for each $Q \in \mathbf{A}$.*

REMARK 3.1. *A detailed analysis of the examples at our disposal and considerations about the problem of prolongation (cf. Remark 4.1) suggest that condition (1) in the above definition should be replaced by the condition that $\mathbf{A}$ is 2-acyclic. As for condition (2) in the definition, it amounts to the vanishing of a cohomology class in the group*

$$H^{0,2}(\mathbf{A}) = \frac{\mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)}{\delta^{1,1}(\mathbf{A} \otimes \mathfrak{a}^*)}.$$

(cf. Remark 4.1 and Appendix B for the the notion of torsion of a linear PDS).

EXAMPLE 3. If $\mathbf{A} \subset \operatorname{Hom}(\mathfrak{a}, \mathfrak{b})$ is involutive (or 2-acyclic) and $\mathfrak{g}$ is the abelian Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, then $\tau(Q) = 0$, for each $Q \in \mathbf{A}$, and $\mathbf{A}$ can

be considered as a tableau over $\mathfrak{g}$. Therefore, the concept of tableau over a Lie algebra is a natural (non-commutative) generalization of the classical notion of involutive (or 2-acyclic) tableau.

EXAMPLE 4. Let $\mathfrak{g}$ be a semisimple Lie algebra with Killing form $\langle\, ,\rangle$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Cartan decomposition. Then

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

Assume that $\operatorname{rank} \mathfrak{g}/\mathfrak{g}_0 = k$ and that $\mathfrak{a}$ be a maximal (k-dimensional) abelian subspace of $\mathfrak{g}_1$. Then $\mathfrak{g}_1 = \mathfrak{a} \oplus \mathfrak{m}$, where

$$\mathfrak{m} = \mathfrak{a}^{\perp} \cap \mathfrak{g}_1$$

Further, let

$$\begin{gathered} (\mathfrak{g}_0)_{\mathfrak{a}} = \{X \in \mathfrak{g}_0 \,:\, [X, \mathfrak{a}] = 0\}\,, \\ (\mathfrak{g}_0)^{\perp}_{\mathfrak{a}} = \{X \in \mathfrak{g} \,:\, \langle X, Y\rangle = 0, \forall\, Y \in (\mathfrak{g}_0)_{\mathfrak{a}}\}\,, \\ \mathfrak{b} := \mathfrak{g}_0 \cap (\mathfrak{g}_0)^{\perp}_{\mathfrak{a}}. \end{gathered}$$

Then, for any regular element $A \in \mathfrak{a}$, the maps

$$\mathrm{ad}_A : \mathfrak{m} \to \mathfrak{b}, \quad \mathrm{ad}_A : \mathfrak{b} \to \mathfrak{m}$$

are vector space isomorphisms and

$$X \in \mathfrak{m} \mapsto -\mathrm{ad}_X \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$$

is injective, hence $\mathfrak{m}$ *can be identified with a linear subspace of* $\mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$.

PROPOSITION 3.1 ([33]). *If* $\mathfrak{g}$ *is a semisimple Lie algebra and* $\mathfrak{a}$, $\mathfrak{b}$, *and* $\mathfrak{m}$ *are defined as above, then* $\mathfrak{m}$, *regarded as a subspace of* $\mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$, *is a tableau over* $\mathfrak{g}$.

DEFINITION 3.2. *The tableau* $\mathfrak{m}$ *is called a* ***Cartan tableau*** *over* $\mathfrak{g}$.

REMARK 3.2. *As already indicated in the introduction, the idea of a tableau over a Lie algebra has its origin in the method of moving frames and is related to the existence of canonical adapted frames along generic submanifolds in homogeneous spaces. The tableaux corresponding to systems of submanifold geometry are constructed on the Lie algebras of the transitive groups of transformations of the ambient spaces, e.g., the Wilczynski–Cartan frame in projective differential geometry (cf. [33]), or the canonical Möbius frame in conformal theory of surfaces (cf. Section 5.3).*

4. Differential systems associated with tableaux over Lie algebras. Let $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ be a tableau over $\mathfrak{g}$ and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We set $Y := G \times \mathbf{A}$ and refer to it as the *configuration space*.

DEFINITION 4.1. *A basis* $(A_1, \dots, A_k, B_1, \dots, B_h, C_1, \dots, C_s)$ *of* $\mathfrak{g}$ *is said adapted to* $\mathbf{A}$ *if*

1. $\mathfrak{a} = \textit{span}\,\{A_1, \dots, A_k\}$,
2. $\operatorname{Im}\mathbf{A} := \sum_{Q\in\mathbf{A}} \operatorname{Im} Q = \textit{span}\,\{B_1, \dots, B_h\}$,
3. $\mathfrak{b} = \textit{span}\,\{B_1, \dots, B_h, C_1, \dots, C_s\}$.

An adapted basis is generic if the flag

$$(0) \subset \textit{span}\,\{A_1\} \subset \cdots \subset \textit{span}\,\{A_1, \dots, A_k\} = \mathfrak{a}$$

is generic with respect to $\mathbf{A}$.

For a generic adapted basis, let

$$(\alpha^1, \dots, \alpha^k, \beta^1, \dots, \beta^h, \gamma^1 \dots, \gamma^s)$$

denote the dual coframe on G. Given a basis

$$Q_\epsilon = Q^j_{\epsilon i} B_j \otimes \alpha^i \quad (\epsilon = 1, \dots m)$$

of the tableau $\mathbf{A}$, Y identifies with $G \times \mathbb{R}^m$ by

$$(g, p^\epsilon Q_\epsilon) \in Y \mapsto (g; p^1, \dots, p^m) \in G \times \mathbb{R}^m.$$

DEFINITION 4.2 ([33]). *The EDS* ***associated with*** $\mathbf{A}$ *is the Pfaffian system* $(\mathcal{I}, \omega)$ *on* Y *generated (as a differential ideal) by the linearly independent 1-forms*

$$\begin{cases} \eta^j := \beta^j - p^\epsilon Q^j_{\epsilon i}\alpha^i, \quad (j = 1, \dots, h), \\ \gamma^1, \dots, \gamma^s, \end{cases}$$

with independent condition $\omega = \alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$.

An immersed submanifold

$$\Phi = (g; p^1, \dots, p^m) : N^k \to G \times \mathbf{A} \cong G \times \mathbb{R}^m.$$

is an **integral manifold** of $(\mathcal{I}, \omega)$ if and only if

1. $(\alpha^1 \wedge \cdots \wedge \alpha^k)_{|N} \neq 0$;
2. $\beta^j = p^\epsilon Q^j_{\epsilon i}\alpha^i$, $j = 1, \dots, h$;
3. $\gamma^1 = \cdots = \gamma^s = 0$.

The main result in the construction is the following.

THEOREM 4.1 ([33]). *Let* $\mathbf{A}$ *be a tableau over a Lie algebra* $\mathfrak{g}$. *Then, the EDS* $(\mathcal{I}, \omega)$ *associated with* $\mathbf{A}$ *is a linear PDS in involution. In particular, the characters of* $\mathbf{A}$ *coincide with the Cartan characters of* $(\mathcal{I}, \omega)$.

REMARK 4.1. *Condition (2) in the definition of a tableau over a Lie algebra (cf. Definition 3.1) tells us that the PDS associated with a tableau over a Lie algebra is linear and with* ***vanishing torsion*** *(cf. Appendix B). This together with the condition that the tableau is 2-acyclic guarantee the existence of a prolongation tower for the associated PDS which can be constructed algebraically from the tableau and its prolongations. The construction of the prolonged systems is a*

direct consequence of the property of the tableau being 2-acyclic and is entirely based on the Spencer cohomology of the tableau. The vanishing of the torsion is needed only at the first step of the construction (cf. Remark B.2). Therefore, the result stated in Theorem 4.1 can be generalized to the following.

THEOREM 4.2 ([35]). *Let $\mathbf{A}$ be a 2-acyclic tableau over a Lie algebra $\mathfrak{g}$. Then, the PDS $(\mathcal{I}, \omega)$ on Y associated with $\mathbf{A}$ admits regular prolongations of any order. Moreover, the construction of prolongations is purely algebraic. The configuration space of the h-prolonged system $(\mathcal{I}^{(h)}, \omega)$ is*

$$Y^{(h)} := G \times (\mathbf{A} \oplus \mathbf{A}^{(1)} \oplus \cdots \oplus \mathbf{A}^{(h)}).$$

If k is the least integer such that $\mathbf{A}^{(k)}$ is involutive, then the k-prolongation $(\mathcal{I}^{(k)}, \omega)$ is in involution and its Cartan characters coincide with that of $\mathbf{A}^{(k)}$.

5. Examples. In this section, we illustrate the construction developed in Section 4 by discussing some examples.

5.1. The PDS associated with an abelian tableau. Let $\mathbf{A} \subset \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^h)$ be an m-dimensional involutive tableau over the (abelian) Lie algebra $\mathfrak{g} = \mathbb{R}^k \oplus \mathbb{R}^h$, spanned by the linearly independent $h \times k$ matrices $Q_\epsilon = (Q^j_{\epsilon i})$.

We call $\mathbf{A}^{(1)}$**-system** the linear, homogeneous, constant coefficient PDE system for maps $P = (p^1, \dots, p^m) : \mathbb{R}^k \to \mathbf{A} \cong \mathbb{R}^m$ defined by the differential inclusion $dP_{|x} \in \mathbf{A}^{(1)}$, for all $x \in \mathbb{R}^k$, where $\mathbf{A}^{(1)}$ is the first prolongation of $\mathbf{A}$. This system can be written

$$\delta^{1,1}(dP) = 0,$$

where $\delta^{1,1}$ is the Spencer coboundary of the tableau $\mathbf{A}$ (recall that $\delta^{1,1} : C^{1,1} = \mathbf{A} \otimes (\mathbb{R}^k)^* \to C^{0,2}$).

LEMMA 5.1. *A map $P : \mathbb{R}^k \to \mathbf{A} \cong \mathbb{R}^m$ is a solution to the $\mathbf{A}^{(1)}$-system if and only if the $\mathbb{R}^h$-valued 1-form*

$$\theta = (\theta^1, \dots, \theta^h) \in \Omega^1(\mathbb{R}^k) \otimes \mathbb{R}^h, \quad \theta^j = p^\epsilon Q^j_{\epsilon a} dx^a,$$

is closed.

As a consequence, we have

COROLLARY 5.1. *Let $P : \mathbb{R}^k \to \mathbf{A} \cong \mathbb{R}^m$ be a solution to the $\mathbf{A}^{(1)}$-system and let $y = (y^1, \dots, y^h)$ be a primitive of θ (i.e., $\theta = dy$). Then*

$$\mathbb{R}^k \ni x \mapsto (x, y(x), P(x)) \in \mathbb{R}^k \oplus \mathbb{R}^h \oplus \mathbb{R}^m$$

is an integral manifold of the PDS $(\mathcal{I}, \omega)$ associated with $\mathbf{A}$. Moreover, every integral manifold of $(\mathcal{I}, \omega)$ arises in this way.

We can conclude that the integral manifolds of $(\mathcal{I}, \omega)$ correspond to the solutions of the $\mathbf{A}^{(1)}$-system. Moreover, $(\mathcal{I}, \omega)$ is in involution (as a differential system) and the Cartan characters coincide with those of the tableau $\mathbf{A}$.

5.2. The PDS associated with a Cartan tableau and the G/G_0-system. Let G/G_0 be a semisimple symmetric space of rank k and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a Cartan decomposition of $\mathfrak{g}$. Let $(A_1, \ldots, A_k)$ be a regular basis for the maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_1$. According to Terng [37], the G/G_0-**system** (or the k-dimensional system associated to G/G_0) is the system of PDEs for maps $F : U \subset \mathfrak{a} \to \mathfrak{m} := \mathfrak{g}_1 \cap \mathfrak{a}^\perp$ defined by

$$\left[A_i, \frac{\partial F}{\partial x^j}\right] - \left[A_j, \frac{\partial F}{\partial x^i}\right] = \left[[A_i, F], [A_j, F]\right],$$

$1 \leq i < j \leq k$, where x^i are the coordinates with respect to $(A_1, \ldots, A_k)$.

LEMMA 5.2. *A map $F : \mathfrak{a} \to \mathfrak{m}$ is a solution of the G/G_0-system if and only if the $\mathfrak{g}$-valued 1-form*

$$\theta = \alpha + [\alpha, F] \in \Omega^1(\mathfrak{a}) \otimes \mathfrak{g},$$

satisfies the Maurer–Cartan equation $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$, where $\alpha = \alpha^i \otimes A_i$ is the tautological 1-form on $\mathfrak{a}$.

COROLLARY 5.2. *Let $F : \mathfrak{a} \to \mathfrak{m}$ be a solution of the G/G_0-system and let $g : \mathfrak{a} \to G$ be a primitive of θ (i.e. a solution to $g^{-1}dg = \theta$). Then*

$$\mathfrak{a} \ni x \mapsto (g(x), F(x)) \in G \times \mathfrak{m}$$

is an integral manifold of the PDS $(\mathcal{I}, \omega)$ on $Y = G \times \mathfrak{m}$ associated with the Cartan tableau $\mathfrak{m} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$. Conversely, any integral manifold of $(\mathcal{I}, \omega)$ arises in this way.

In conclusion, the integral manifolds of the PDS $(\mathcal{I}, \omega)$ associated with the Cartan tableau $\mathfrak{m} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ are given by the solutions of the corresponding G/G_0-system. Moreover, $(\mathcal{I}, \omega)$ is in involution and its Cartan characters coincide with those of the tableau $\mathfrak{m}$ (i.e., $s_1 = n$, $s_j = 0$, $j = 2, \ldots, n$). In particular, the general solution depends on n functions in one variable.

REMARK 5.1. *If $(A_1, \ldots, A_n)$ is a basis of $\mathfrak{a}$, $(x^1, \ldots, x^n)$ the corresponding coordinates, and $F = (F^1, \ldots, F^{m-n}) : U \subset \mathfrak{a} \to \mathfrak{b}$, then the G/G_0-system can be written*

$$B^a_{\alpha,i} \frac{\partial F^\alpha}{\partial x^j} - B^a_{\alpha,j} \frac{\partial F^\alpha}{\partial x^i} = \Phi^a_{ij},$$

where the coefficients $B^a_{\alpha i}$ are constant and Φ_{ij} are analytic functions.

In general, the PDS associated to a tableau over a Lie algebra corresponds to the nonlinear system of equations

$$B^a_{\alpha,i} \frac{\partial F^\alpha}{\partial x^j} - B^a_{\alpha,j} \frac{\partial F^\alpha}{\partial x^i} + B^a_{\alpha,\beta} \left(\frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} - \frac{\partial F^\alpha}{\partial x^j} \frac{\partial F^\beta}{\partial x^i} \right) = \Phi^a_{ij}.$$

5.3. Old and new involutive systems in conformal surface theory. In this section we discuss some old and new involutive systems/tableaux arising in conformal geometry of surfaces. We start by recalling some preliminary material. Consider Minkowski 5-space $\mathbb{R}^{4,1}$ with linear coordinates $x^0, \dots, x^4$ and Lorentz scalar product given by the quadratic form

$$\langle x, x\rangle = -x^0x^4 + (x^1)^2 + (x^2)^2 + (x^3)^2 = \eta_{ij}x^ix^j. \tag{5.1}$$

Classically, the Möbius space S^3 (conformal 3-sphere) is realized as the projective quadric $\{[x] \in \mathbb{RP}^4 : \langle x, x\rangle = 0\}$. Accordingly, S^3 inherits a natural conformal structure and the identity component $G \cong \mathrm{SO}_0(4,1)$ of the pseudo-orthogonal group of (5.1) acts transitively on S^3 as group of orientation-preserving, conformal transformations (see [5]). The Maurer–Cartan form of G will be denoted by $\omega = (\omega^i_j)$.

Let $f : U \subset \mathbb{R}^2 \to S^3$ be an umbilic free, conformal immersion. A *Möbius frame field* along f is a map $g = (g_0, \dots, g_4) : U \to G$ such that $f(p) = [g_0(p)]$, for all $p \in U$. According to [5], there exists a canonical Möbius frame field[2] $g : U \to G$ along f such that its Maurer–Cartan form $\beta = (\beta^i_j) = g^*\omega$ takes the form

$$\begin{pmatrix} -2q_2\beta^1_0 + 2q_1\beta^2_0 & p_1\beta^1_0 + p_2\beta^2_0 & -p_2\beta^1_0 + p_3\beta^2_0 & 0 & 0 \\ \beta^1_0 & 0 & -q_1\beta^1_0 - q_2\beta^2_0 & -\beta^1_0 & p_1\beta^1_0 + p_2\beta^2_0 \\ \beta^2_0 & q_1\beta^1_0 + q_2\beta^2_0 & 0 & \beta^2_0 & -p_2\beta^1_0 + p_3\beta^2_0 \\ 0 & \beta^1_0 & -\beta^2_0 & 0 & 0 \\ 0 & \beta^1_0 & \beta^2_0 & 0 & 2q_2\beta^1_0 - 2q_1\beta^2_0 \end{pmatrix}$$

with $\beta^1_0 \wedge \beta^2_0 > 0$. The smooth functions q_1, q_2, p_1, p_2, p_3 form a complete system of conformal invariants for f and satisfy the following structure equations

$$d\beta^1_0 = -q_1\beta^1_0 \wedge \beta^2_0, \quad d\beta^2_0 = -q_2\beta^1_0 \wedge \beta^2_0, \tag{5.2}$$
$$dq_1 \wedge \beta^1_0 + dq_2 \wedge \beta^2_0 = (1 + p_1 + p_3 + {q_1}^2 + {q_2}^2)\beta^1_0 \wedge \beta^2_0, \tag{5.3}$$
$$dq_2 \wedge \beta^1_0 - dq_1 \wedge \beta^2_0 = -p_2\beta^1_0 \wedge \beta^2_0, \tag{5.4}$$
$$dp_1 \wedge \beta^1_0 + dp_2 \wedge \beta^2_0 = (4q_2p_2 + q_1(3p_1 + p_3))\beta^1_0 \wedge \beta^2_0, \tag{5.5}$$
$$dp_2 \wedge \beta^1_0 - dp_3 \wedge \beta^2_0 = (4q_1p_2 - q_2(p_1 + 3p_3))\beta^1_0 \wedge \beta^2_0. \tag{5.6}$$

5.3.1. The Möbius tableau. The existence of a canonical Möbius frame field suggests the following construction. Let $(\alpha^1, \alpha^2, \beta^1, \dots, \beta^4, \gamma^1, \dots, \gamma^4)$ be the basis of $\mathfrak{g}^*$ defined by

$$\begin{cases} \alpha^1 = \omega^1_0, \quad \alpha^2 = \omega^2_0, \quad \beta^1 = \omega^0_0, \quad \beta^2 = \omega^0_1, \quad \beta^3 = \omega^0_2, \quad \beta^4 = \omega^2_1, \\ \gamma^1 = \omega^0_3, \quad \gamma^2 = \omega^3_0, \quad \gamma^3 = \omega^1_0 - \omega^3_1, \quad \gamma^4 = \omega^2_0 + \omega^3_2. \end{cases}$$

Next, let $(A_1, A_2, B_1, \dots, B_4, C_1, \dots, C_4)$ be its dual basis and set

$$\mathfrak{a} = \mathrm{span}\,\{A_1, A_2\}, \quad \mathfrak{b} = \mathrm{span}\,\{B_1, \dots, B_4\}.$$

[2] If g is a canonical frame, any other canonical frame is given by $(g_0, -g_1, -g_2, g_3, g_4)$.

Consider the 5-dimensional subspace $\mathbf{M} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ consisting of all elements $Q(q, p)$ of the form

$$\begin{aligned} Q(q,p) = q_1(B_4 \otimes \alpha^1 + 2B_1 \otimes \alpha^2) + q_2(-2B_1 \otimes \alpha^1 + B_4 \otimes \alpha^2) \\ + p_1 B_2 \otimes \alpha^1 + p_2(-B_3 \otimes \alpha^1 + B_2 \otimes \alpha^2) + p_3 B_3 \otimes \alpha^2, \end{aligned}$$

where $q = (q_1, q_2) \in \mathbb{R}^2$, $p = (p_1.p_2, p_3) \in \mathbb{R}^3$. A direct computation yields the following.

LEMMA 5.3. *The subspace* $\mathbf{M}$ *is a tableau over* $\mathfrak{g} \cong \mathfrak{so}(4, 1)$.

The PDS associated with the tableau $\mathbf{M}$, referred to as the Möbius system, is the PDS on $Y = G \times \mathbf{M} \cong G \times \mathbb{R}^5$ generated by the 1-forms $\gamma^1, \dots, \gamma^4, \eta^1, \dots, \eta^4$, where

$$\begin{cases} \eta^1 = \beta^1 + 2q_2\omega_0^1 - 2q_1\omega_0^2, \quad \eta^2 = \beta^2 - p_1\omega_0^1 - p_2\omega_0^2, \\ \eta^3 = \beta^3 + p_2\omega_0^1 - p_3\omega_0^2, \quad \eta^4 = \beta^4 - q_1\omega_0^1 - q_2\omega_0^2, \end{cases}$$

with independence condition $\omega_0^1 \wedge \omega_0^2 \neq 0$. The integral manifolds of the Möbius system are the 2-dimensional submanifolds

$$(g; q, p) : M^2 \to G \times \mathbf{M} \cong G \times \mathbb{R}^5$$

such that:

- $f = [g_0] \to S^3$ is an umbilic free, conformal immersion;
- $g : M^2 \to G$ is a canonical Möbius frame along f;
- q_1, q_2, p_1, p_2, $p_3 : M^2 \to \mathbb{R}$ are the conformal invariants of f.

5.3.2. Willmore surfaces. *Willmore immersions* are defined as extremals for the Willmore functional $\int (H^2 - K)dA$ (H the mean curvature, K the Gauss curvature). They are characterized by the Euler–Lagrange equation

$$\Delta H + 2H(H^2 - K) = 0,$$

which expressed in terms of the conformal invariants is equivalent to the equation $p_1 = p_3$ (cf. [5, 30, 39]). Willmore surfaces can be seen as integral manifolds of the Möbius system restricted to the submanifold of Y given by

$$Y_W = \{Q(q, p) \in Y \,|\, p_1 = p_3\}.$$

Now, it is easy to check that the subspace

$$\mathbf{M}_W := \{Q(q, p) \in \mathbf{M} \,|\, p_1 = p_3\}$$

defines a 4-dimensional tableau over $\mathfrak{g} \cong \mathfrak{so}(4, 1)$ with characters $s_0 = 8$, $s_1 = 4$, $s_2 = 0$, and that Y_W is the configuration space of $\mathbf{M}_W$. Observe also that the restriction to Y_W of the Möbius system is exactly the PDS associated with $\mathbf{M}_W$.

5.3.3. Other classes of surfaces. More generally, one could consider the class of surfaces whose invariant functions p_1 and p_3 satisfy a linear relation, that is, are expressed by $p_1(t) = t\cos a + b_1$, $p_2(t) = t\sin a + b_2$, for real constants a, b_1, b_2. This class includes Willmore surfaces as special examples and corresponds to the 4-dimensional affine tableau

$$\mathbf{M}_{(a,b_1,b_2)} = \{Q(q, p_1(t), p_2, p_3(t)) \,|\, t, p_2 \in \mathbb{R}, q \in \mathbb{R}^2\}.$$

Also in this case, an algebraic, direct computation shows that $\mathbf{M}_{(a,b_1,b_2)}$ is involutive. Therefore, by the construction developed in the previous section, the associated PDS is in involution. Its Cartan characters are $s_0 = 8$, $s_1 = 4$, $s_2 = 0$.

REMARK 5.2. *Similar arguments have been used in connection with the study of surfaces in projective differential geometry [33]. The same approach can also be used to discuss surface theory in the framework of Laguerre geometry [27] and other classical geometries.*

6. Further developments.

6.1. Continue the program, initiated with the study of several classes of integrable surfaces in projective differential geometry [33], of identifying the geometry associated to a given tableau/system, i.e., find submanifolds in some homogeneous space whose integrability conditions are given by the PDS associated with the given tableau.

6.2. Study the algebraic structure of tableaux over Lie algebras to understand when a tableau generates an integrable geometry.

6.3. Study the Cauchy problem for the associated systems (cf. [31, 34]).

6.4. Analyze the characteristic cohomology of a tableau over a Lie algebra, its geometric interpretations, and its relations with the characteristic cohomology of Bryant–Griffiths [8, 9].

APPENDIX

A. The Spencer complex. [cf. [7]]

Retaining the notation of Section 2, identify the symmetric product $S^q(\mathfrak{a}^*)$ with the space of homogeneous polynomials of degree q on $\mathfrak{a}$. For each $v \in \mathfrak{a}$, let δ_v be the map of $\mathfrak{b} \otimes S^q(\mathfrak{a}^*) \to \mathfrak{b} \otimes S^{q-1}(\mathfrak{a}^*)$ given by partial differentiation w.r.t. v. Let $v_1, \dots, v_n$ be a basis of $\mathfrak{a}$, and $v^1, \dots, v^n$ its dual basis.

The operator

$$\mathfrak{b} \otimes S^q(\mathfrak{a}^*) \otimes \Lambda^p(\mathfrak{a}^*) \xrightarrow{\delta^{q,p}} \mathfrak{b} \otimes S^{q-1}(\mathfrak{a}^*) \otimes \Lambda^{p+1}(\mathfrak{a}^*)$$

given by

$$\delta^{q,p}\xi := \sum \delta_{v_i}\xi \wedge v^i$$

($\delta^{0,p} = 0$, for $p \geq 0$) is independent of the basis, $\delta^2 = 0$, and the sequence of the corresponding bigraded complex is exact except when $q = 0$ and $p = 0$.

Let $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ be a tableau with prolongations $\mathbf{A}^{(h)}$, $h \geq 0$. Consider the sequence of spaces

$$C^{q,p}(\mathbf{A}) := \mathbf{A}^{(q-1)} \otimes \Lambda^p(\mathfrak{a}^*),$$

for integers $q \geq 0$ and $0 \leq p \leq n$. Note that since $\mathbf{A}^{(q-1)} \subset \mathfrak{b} \otimes S^q(\mathfrak{a}^*)$, the space $C^{q,p}(\mathbf{A})$ is a subspace of $\mathfrak{b} \otimes S^q(\mathfrak{a}^*) \otimes \Lambda^p(\mathfrak{a}^*)$. We have

$$\delta C^{q,p}(\mathbf{A}) \subset C^{q-1,p+1}(\mathbf{A}),$$

but the sequence

$$C^{q+1,p-1}(\mathbf{A}) \xrightarrow{\delta^{q+1,p-1}} C^{q,p}(\mathbf{A}) \xrightarrow{\delta^{q,p}} C^{q-1,p+1}(\mathbf{A})$$

is no longer exact for all p and q. The associated cohomology groups

$$H^{q,p}(\mathbf{A}) := Z^{q,p}(\mathbf{A})/B^{q,p}(\mathbf{A})$$

are called the *Spencer groups* of $\mathbf{A}$, where $B^{q,p}(\mathbf{A}) = \mathrm{Im}(\delta^{q+1,p-1})$ and $Z^{q,p}(\mathbf{A}) = \mathrm{Ker}\,(\delta^{q,p})$. Notice that $Z^{0,p}(\mathbf{A}) = \mathfrak{b} \otimes \Lambda^p(\mathfrak{a}^*)$, for all $p \geq 0$, and $Z^{q,1}(\mathbf{A}) = \mathbf{A}^{(q)}$, for all $q \geq 1$.

A significant result in the subject is that the vanishing of the $H^{q,p}$ is equivalent to involutiveness.

THEOREM A.1 ([20]). *A tableau $\mathbf{A}$ is involutive if and only if $H^{q,p}(\mathbf{A})$ is zero, for all $q \geq 1$ and $p \geq 0$.*

A weaker condition than involutiveness is the following.

DEFINITION A.1. *A tableau $\mathbf{A}$ is called 2-acyclic if $H^{q,2}(\mathbf{A}) = (0)$, for all $q \geq 1$.*

Another way of formulating the condition

$$H^{q,p}(\mathbf{A}) = (0), \quad \text{for all} \quad q \geq 1,\, p \geq 0$$

is that the sequences

$$0 \to \mathbf{A}^{(k)} \xrightarrow{\delta} \mathbf{A}^{(k-1)} \otimes \mathfrak{a}^* \to \cdots \xrightarrow{\delta} \mathbf{A} \otimes \Lambda^{k-1}(\mathfrak{a}^*) \to$$

$$\cdots \xrightarrow{\delta} \mathfrak{b} \otimes \Lambda^k(\mathfrak{a}^*) \to \frac{\mathfrak{b} \otimes \Lambda^k(\mathfrak{a}^*)}{\delta\left(\mathbf{A} \otimes \Lambda^{k-1}(\mathfrak{a}^*)\right)} \to 0$$

are exact for all k. In particular, we have

$$H^{0,k}(\mathbf{A}) = \frac{\mathrm{Ker}\,(\delta^{0,k})}{\mathrm{Im}\,(\delta^{1,k-1})} = \frac{\mathfrak{b} \otimes \Lambda^k(\mathfrak{a}^*)}{\delta\left(\mathbf{A} \otimes \Lambda^{k-1}(\mathfrak{a}^*)\right)}.$$

B. Torsion of a Pfaffian systems. [cf. [7]]

Retaining the notation of Example 2, let $(\mathcal{I},\omega)$ be a Pfaffian system. An admissible integral element $E \in V_n(\mathcal{I},\omega)_{|x}$ is given by

$$\theta^a = 0, \quad \pi^\epsilon = p^\epsilon_i \omega^i,$$

where the fiber coordinates p^ϵ_i satisfy

$$A^a_{\epsilon\, j}(x) p^\epsilon_i - A^a_{\epsilon\, i}(x) p^\epsilon_j + c^a_{i\, j}(x) = 0.$$

Under a change of coframe

$$\tilde{\theta}^a = \theta^a, \quad \tilde{\omega}^i = \omega^i, \quad \tilde{\pi}^\epsilon = \pi^\epsilon - p^\epsilon_i \omega^i \tag{B.1}$$

the numbers $c^a_{i\, j}(x)$ transform to

$$\tilde{c}^a_{ij}(x) = A^a_{\epsilon\, j}(x) p^\epsilon_i - A^a_{\epsilon\, i}(x) p^\epsilon_j + c^a_{i\, j}(x).$$

This defines an equivalence relation

$$\tilde{c}^a_{ij}(x) \sim c^a_{ij}(x).$$

DEFINITION B.1. *The equivalence class $[c^a_{ij}(x)]$ is called the **torsion** of $(\mathcal{I},\omega)$.*

LEMMA B.1. *The torsion of $(\mathcal{I},\omega)$ lives in*

$$H^{0,2}(\mathbf{A}) = \frac{W \otimes \Lambda^2(V^*)}{\delta^{1,1}\left(\mathbf{A} \otimes V^*\right)} = \frac{\mathrm{Ker}\,(\delta^{0,2})}{\mathrm{Im}\,(\delta^{1,1})}.$$

Proof. If $Q = p^\epsilon_j A^a_{\epsilon\, i} \frac{\partial}{\partial\theta^a} \otimes \omega^i \otimes \omega^j \in \mathbf{A} \otimes V^*$, then

$$\delta^{1,1}\left(Q\right) = \sum_{i<j} \left(p^\epsilon_j A^a_{\epsilon\, i} - p^\epsilon_i A^a_{\epsilon\, j}\right) \frac{\partial}{\partial\theta^a} \otimes \omega^i \wedge \omega^j.$$

According to the transformation rule (B.1) of the c^a_{ij} under a coframe change, the cocycle

$$\frac{1}{2} c^a_{i\, j}(x) \frac{\partial}{\partial\theta^a} \otimes \omega^i \wedge \omega^j \in C^{0,2}(\mathbf{A})$$

gives a class in $H^{0,2}(\mathbf{A})$. ☐

REMARK B.1. *The vanishing of the torsion is a necessary and sufficient condition for the existence of an integral element over x.*

LEMMA B.2. *The torsion of $(\mathcal{I}^{(1)},\omega)$ lives in the vector spaces*

$$H^{0,2}(\mathbf{A}^{(1)}) \cong H^{1,2}(\mathbf{A}).$$

We also recall the following.
LEMMA B.3.

$$H^{q,p}(\mathbf{A}^{(1)}) \cong H^{q+1,p}(\mathbf{A}), \quad q \geq 1.$$

REMARK B.2. *Thus, the involutiveness of the tableau A associated to $(\mathcal{I}, \omega)$ implies both the involutiveness of the tableau $A^{(1)}$ and the vanishing of torsion of the prolonged system $(\mathcal{I}^{(1)}, \omega)$.*

REFERENCES

[1] M. A. Akivis, V. V. Goldberg, *Projective differential geometry of submanifolds*, North-Holland Mathematical Library, 49, North-Holland Publishing Co., Amsterdam, 1993.

[2] G. Bol, *Projektive Differentialgeometrie, 2. Teil*, Studia mathematica, B. IX, Vandenhoeck & Ruprecht, Göttingen, 1954.

[3] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, B. 3, bearbeitet von G. Thomsen, J. Springer, Berlin, 1929.

[4] M. Brück, X. Du, J. Park, and C.-L. Terng, Submanifold geometry of real Grassmannian systems, The Memoirs, vol. 155, AMS, **735** (2002), 1–95.

[5] R. Bryant, A duality theorem for Willmore surfaces, *J. Differential Geom.* **20** (1984), no. 1, 23–53.

[6] R. L. Bryant, Lectures given at MSRI "Integrable system seminar", 2003, unpublished notes.

[7] R. L. Bryant, S.-S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, 18, Springer-Verlag, New York, 1991.

[8] R. L. Bryant, P. A. Griffiths, Characteristic cohomology of differential systems, I: General theory, *J. Amer. Math. Soc.* **8** (1995), 507–596.

[9] ———, Characteristic cohomology of differential systems, II: Conservations laws for a class of parabolic equations, *Duke Math. J.* **78** (1995), 531–676.

[10] É. Cartan, *Sur le problème général de la déformation*, C. R. Congrés Strasbourg (1920), 397–406; or *Oeuvres Complètes*, III 1, 539–548.

[11] É. Cartan, Sur la déformation projective des surfaces, *Ann. Scient. Éc. Norm. Sup. (3)* **37** (1920), 259–356; or *Oeuvres Complètes*, III 1, 441–538.

[12] É. Cartan, *Les systèmes différentielles extérieurs et leurs applications géométriques*, Hermann, Paris, 1945.

[13] E. Ferapontov, Lie sphere geometry and integrable systems, *Tohoku Math. J.* **52** (2000), 199–233.

[14] ———, Integrable systems in projective differential geometry, *Kyushu J. Math.* **54** (2000), 183–215.

[15] ———, The analogue of Wilczynski's projective frame in Lie sphere geometry: Lie-applicable surfaces and commuting Schrödinger operators with magnetic fields, *Internat. J. Math.* **13** (2002), 956–986.

[16] D. Ferus, F. Pedit, Curved flats in symmetric spaces, *Manuscripta Math.* **91** (1996), 445–454.

[17] S. P. Finikov, *Projective Differential Geometry*, Moscow, Leningrad, 1937.

[18] H. Goldschmidt, Existence theorems for analytic linear partial differential equations, *Ann. of Math. (2)* **86** (1967), 246–270.

[19] H. Goldschmidt, Integrability criteria for systems of nonlinear partial differential equations, *J. Differential Geom.* **1** (1967), 269–307.

[20] V. W. Guillemin, S. Sternberg, An algebraic model of transitive differential geometry, *Bull. Amer. Math. Soc.* **70** (1964), 16–47.

[21] T. A. Ivey, J. M. Landsberg, *Cartan for beginners: differential geometry via moving frames and*

exterior differential systems, Graduate Studies in Mathematics, 61, American Mathematical Society, Providence, RI, 2003.
[22] G. R. Jensen, Deformation of submanifolds of homogeneous spaces, *J. Differential Geom.* **16** (1981), 213–246.
[23] ———, Higher order contact of submanifolds of homogeneous spaces. Lecture Notes in Mathematics, vol. 610. Springer-Verlag, Berlin-New York, 1977.
[24] M. Kuranishi, On E. Cartan's prolongation theorem of exterior differential systems, *Amer. J. Math.* **79** (1957), 1–47.
[25] E. Musso, Deformazione di superfici nello spazio di Möbius, *Rend. Istit. Mat. Univ. Trieste* **27** (1995), 25–45.
[26] E. Musso, L. Nicolodi, *On the equation defining isothermic surfaces in Laguerre geometry*, New developments in Differential geometry, Budapest 1996, Kluver Academic Publishers, 285-294.
[27] ———A variational problem for surfaces in Laguerre geometry, *Trans. Amer. Math. Soc.* **348** (1996), 4321–4337.
[28] ———, Isothermal surfaces in Laguerre geometry, *Boll. Un. Mat. Ital. (7) II-B, Suppl. fasc. 2,* **11** (1997), 125–144.
[29] ———, Willmore canal surfaces in Euclidean space, *Rend. Istit. Mat. Univ. Trieste* **31** (1999), 1–26.
[30] ———, Darboux transforms of Dupin surfaces, *Banach Center Publ.* **57** (2002), 135–154.
[31] ———, On the Cauchy problem for the integrable system of Lie minimal surfaces, *J. Math. Phys.* **46** (2005), no. 11, 3509-3523.
[32] ———, Deformation and applicability of surfaces in Lie sphere geometry, *Tohoku Math. J.* **58** (2006), no. 2, 161-187; preprint available as `math.DG/0408009`.
[33] ———, Tableaux over Lie algebras, integrable systems, and classical surface theory, *Comm. Anal. Geom.* **14** (2006), no. 3, 475-496; preprint available as `math.DG/0412169`.
[34] ———, A class of overdetermined systems defined by tableaux: Involutiveness and Cauchy problem, *Phys. D* **229** (2007), no. 1, 35–42; preprint available as `math.DG/0602676`.
[35] ———, in preparation.
[36] P. J. Olver, *Equivalence, invariants, and symmetry*, Cambridge University Press, Cambridge, 1995.
[37] C.-L. Terng, Soliton equations and differential geometry, *J. Differential Geometry* **45** (1997), 407–445.
[38] C.-L. Terng, E. Wang, Curved Flats, exterior differential systems and conservation laws, in: Complex, Contact and Symmetric Manifolds (in honor of L. Vanhecke), O. Kowalski; E. Musso; D. Perrone (Eds.), Progress in Mathematics, Vol. 234, Birkhäuser, 2005, 235–254.
[39] T. J. Willmore, *Riemannian geometry*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.